

GEOMETRY

impares vel pares multitudine, cum hæc, ut dixi, loco ad quatuor lineas respondeant, nullum igitur posuerunt ita ut linea nota sit, &c.^[34]

The question, then, the solution of which was begun by Euclid and carried farther by Apollonius, but was completed by no one, is this:

Having three, four or more lines given in position, it is first required to find a point from which as many other lines may be drawn, each making a given angle with one of the given lines, so that the rectangle of two of the lines so drawn shall bear a given ratio to the square of the third (if there be only three); or to the rectangle of the other two (if there be four), or again, that the parallelepiped^[35] constructed upon three shall bear a given ratio to that upon the other two and any given line (if there be five), or to the parallelepiped upon the other three (if there be six); or (if there be seven) that the product obtained by multiplying four of them together shall bear a given ratio to the product of the other three, or (if there be eight) that the product of four of them shall bear a given ratio to the product of the other four. Thus the question admits of extension to any number of lines.

Then, since there is always an infinite number of different points satisfying these requirements, it is also required to discover and trace the curve containing all such points.^[36] Pappus says that when there are only three or four lines given, this line is one of the three conic sections, but he does not undertake to determine, describe, or explain the nature of the line required^[37] when the question involves a greater number of lines. He only adds that the ancients recognized one of them which they had shown to be useful, and which seemed the sim-

^[34] This rather obscure passage may be translated as follows: "For in this are agreed those who formerly interpreted these things (that the dimensions of a figure cannot exceed three) in that they maintain that a figure that is contained by these lines is not comprehensible in any way. This is permissible, however, both to say and to demonstrate generally by this kind of proportion, and in this manner: If from any point straight lines be drawn making given angles with straight lines given in position; and if there be given a ratio compounded of them, that is the ratio that one of the lines drawn has to one, the second has to a second, the third to a third, and so on to the given line if there be seven lines, or, if there be eight lines, of the last to a last, the point lies on the lines that are given in position. And similarly, whatever may be the odd or even number, since these, as I have said, correspond in position to the four lines; therefore they have not set forth any method so that a line may be known." The meaning of the passage appears from that which follows in the text.

^[35] That is, continued product.

^[36] It is here that the essential feature of the work of Descartes may be said to begin.

^[37] See line 19 on the opposite page.

lelepipedé composé des deux qui restent, & d'une autre ligne donnée. Ou s'il y en a fix, que le parallelepipedé composé de trois ait la proportion donnée avec le parallelepipedé des trois autres. Ou s'il y en a sept, que ce qui se produist lorsqu'on en multiplie quatre l'une par l'autre, ait la raison donnée avec ce qui se produist par la multiplication des trois autres, & encore d'une autre ligne donnée; Ou s'il y en a huit, que le produit de la multiplication de quatre ait la proportion donnée avec le produit des quatre autres. Et ainsi cete question se peut estendre a tout autre nombre de lignes. Puis a cause qu'il y a tousiours une infinité de diuers poins qui peuvent satisfaire a ce qui est icy demandé, il est aussi requis de connoistre, & de tracer la ligne, dans laquelle ils doiuent tous se trouuer. & Pappus dit que lorsqu'il n'y a que trois ou quatre lignes droites données, c'est en une des trois sections coniques. mais il n'entreprend point de la determiner, ny de la descrire. non plus que d'expliquer celles ou tous ces poins se doiuent trouuer, lorsque la question est proposée en vn plus grand nombre de lignes. Seulement il aiouste que les anciens en auoient imaginé une qu'ils monstroient y estre vtile, mais qui sembloit la plus manifeste, & qui n'estoit pas toutefois la premiere. Ce qui m'a donné occasion d'essayer si par la methode dont ie me fers on peut aller aussi loin qu'ils ont esté.

Et premierement i'ay connu que cete question n'estant proposée qu'en trois, ou quatre, ou cinq lignes, on peut toujours trouuer les poins cherchés par la Geometrie simple; c'est a dire en ne se seruant que de la reigle & du

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compas,

compas, ny ne faisant autre chose, que ce qui a desia esté dit; excepté seulement lorsqu'il y a cinq lignes données, si elles sont toutes paralleles. Auquel cas, comme aussy lorsque la question est proposée en six, ou 7, ou 8, ou 9 lignes, on peut toufiours trouuer les poins cherchés par la Geometrie des solides; c'est a dire en y employant quelqu'vne des trois sections coniques. Excepté seulement lorsqu'il y a neuf lignes données, si elles sont toutes paralleles. Auquel cas derechef, & encore en 10, 11, 12, ou 13 lignes on peut trouuer les poins cherchés par le moyen d'vne ligne courbe qui soit dvn degré plus composée que les sections coniques. Excepté en treize si elles sont toutes paralleles, auquel cas, & en quatorze, 15, 16, & 17 il y faudra employer vne ligne courbe encore dvn degré plus composée que la precedente & ainsi a l'infini.

Puis iay trouué aussy, que lorsqu'il ny a que trois ou quatre lignes données, les poins cherchés se rencontrent tous, non seulement en l'vne des trois sections coniques, mais quelquefois aussy en la circonference dvn cercle, ou en vne ligne droite. Et que lorsqu'il y en a cinq, ou six, ou sept, ou huit, tous ces poins se rencontrent en quelque vne des lignes, qui sont dvn degré plus composées que les sections coniques, & il est impossible d'en imaginer aucune qui ne soit vtile a cete question; mais ils peuvent aussi derechef se rencontrer en vne section conique, ou en vn cercle, ou en vne ligne droite. Et s'il y en a neuf, ou 10, ou 11, ou 12, ces poins se rencontrent en vne ligne, qui ne peut estre que dvn degré plus composée que les precedentes; mais toutes celles qui

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plest, and yet was not the most important.^[38] This led me to try to find out whether, by my own method, I could go as far as they had gone.^[39]

First, I discovered that if the question be proposed for only three, four, or five lines, the required points can be found by elementary geometry, that is, by the use of the ruler and compasses only, and the application of those principles that I have already explained, except in the case of five parallel lines. In this case, and in the cases where there are six, seven, eight, or nine given lines, the required points can always be found by means of the geometry of solid loci,^[40] that is, by using some one of the three conic sections. Here, again, there is an exception in the case of nine parallel lines. For this and the cases of ten, eleven, twelve, or thirteen given lines, the required points may be found by means of a curve of degree next higher than that of the conic sections. Again, the case of thirteen parallel lines must be excluded, for which, as well as for the cases of fourteen, fifteen, sixteen, and seventeen lines, a curve of degree next higher than the preceding must be used; and so on indefinitely.

Next, I have found that when only three or four lines are given, the required points lie not only all on one of the conic sections but sometimes on the circumference of a circle or even on a straight line.^[41]

When there are five, six, seven, or eight lines, the required points lie on a curve of degree next higher than the conic sections, and it is impossible to imagine such a curve that may not satisfy the conditions of the problem; but the required points may possibly lie on a conic section, a circle, or a straight line. If there are nine, ten, eleven, or twelve lines, the required curve is only one degree higher than the preceding, but any such curve may meet the requirements, and so on to infinity.

^[38] See lines 5-10 from the foot of page 23.

^[39] Descartes gives here a brief summary of his solution, which he amplifies later.

^[40] This term was commonly applied by mathematicians of the seventeenth century to the three conic sections, while the straight line and circle were called plane loci, and other curves linear loci. See Fermat, *Isagoge ad Locos Planos et Solidos*, Toulouse, 1679.

^[41] Degenerate or limiting forms of the conic sections.

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Finally, the first and simplest curve after the conic sections is the one generated by the intersection of a parabola with a straight line in a way to be described presently.

I believe that I have in this way completely accomplished what Pappus tells us the ancients sought to do, and I will try to give the demonstration in a few words, for I am already wearied by so much writing.

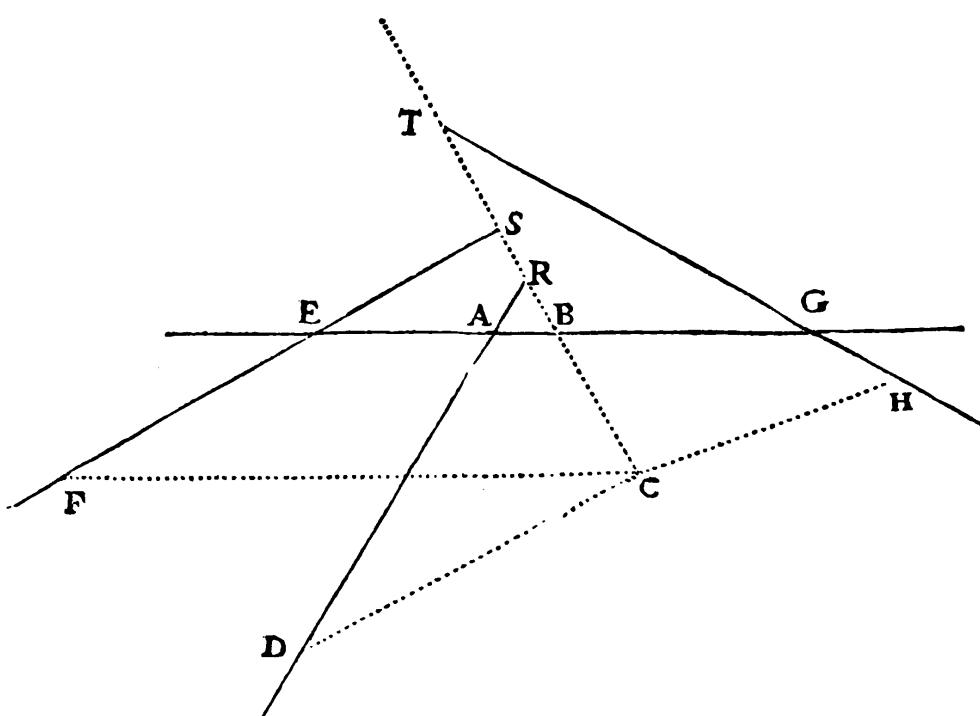
Let AB, AD, EF, GH, ... be any number of straight lines given in position,^[42] and let it be required to find a point C, from which straight lines CB, CD, CF, CH, ... can be drawn, making given angles CBA, CDA, CFE, CHG, ... respectively, with the given lines, and

[42] It should be noted that these lines are given in position but not in length. They thus become lines of reference or coördinate axes, and accordingly they play a very important part in the development of analytic geometry. In this connection we may quote as follows: "Among the predecessors of Descartes we reckon, besides Apollonius, especially Vieta, Oresme, Cavalieri, Roberval, and Fermat, the last the most distinguished in this field; but nowhere, even by Fermat, had any attempt been made to refer several curves of different orders simultaneously to one system of coördinates, which at most possessed special significance for one of the curves. It is exactly this thing which Descartes systematically accomplished." Karl Fink, *A Brief History of Mathematics*, trans. by Beman and Smith, Chicago, 1903, p. 229.

Heath calls attention to the fact that "the essential difference between the Greek and the modern method is that the Greeks did not direct their efforts to making the fixed lines of a figure as few as possible, but rather to expressing their equations between areas in as short and simple a form as possible." For further discussion see D. E. Smith, *History of Mathematics*, Boston, 1923-25, Vol. II, pp. 316-331 (hereafter referred to as Smith).

qui sont dvn degré plus composées y peuvent seruir, & ainsi a l'infini.

Au reste la premiere, & la plus simple de toutes aprés les sections coniques, est celle qu'on peut descrire par l'intersection d'une Parabole, & d'une ligne droite, en la façon qui sera tantost expliquée. En sorte que ie pense auoir entierement satisfait a ce que Pappus nous dit auoir esté cherché en cecy par les anciens. & ie tascheray d'en mettre la demonstration en peu de mots. car il m'ennuie desia d'en tant ecrire.



Soient A B, A D, E F, G H, &c. plusieurs lignes données par position, & qu'il faille trouuer vn point, comme C, duquel ayant tiré d'autres lignes droites sur les données, comme C B, C D, C F, & C H, en sorte que les angles C B A, C D A, C F E, C H G, &c. soient donnés,

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& que ce qui est produit par la multiplication d'une partie de ces lignes, soit égal à ce qui est produit par la multiplication des autres, ou bien qu'ils aient quelque autre proportion donnée, car cela ne rend point la question plus difficile.

Comment on doit poser les termes pour venir à l'E. question en cet exemple. Premièrement ie suppose la chose comme desia faite, & pour me demeuler de la cōfusion de toutes ces lignes, ie considere l'vne des données, & l'vne de celles qu'il faut trouuer, par exemple A B, & C B, comme les principales, & ausquelles ie tasche de rapporter ainsi toutes les autres. Que le segment de la ligne A B, qui est entre les poins A & B, soit nommé x . & que B C soit nommé y . & que toutes les autres lignes données soient prolongées, iusques a ce qu'elles coupent ces deux, aussi prolongées s'il est besoin, & si elles ne leur sont point parallèles. comme vous voyez icy qu'elles coupent la ligne A B aux poins A, E, G, & B C aux poins R, S, T. Puis a cause que tous les angles du triangle A R B sont donnés, la proportion, qui est entre les costés A B, & B R, est aussi donnée, & ie la pose comme de z à b , de façon qu' A B étant x , R B sera $\frac{bx}{z}$, & la toute C R sera $y + \frac{bx}{z}$, à cause que le point B tombe entre C & R; car si R tomboit entre C & B, C R seroit $y - \frac{bx}{z}$; & si C tomboit entre B & R, C R seroit $-y + \frac{bx}{z}$. Tout de mesme les trois angles du triangle D R C sont donnés, & par consequent aussi la proportion qui est entre les costés C R, & C D, que ie pose comme de z à c : de façon que C R étant $y + \frac{bx}{z}$, C D

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such that the product of certain of them is equal to the product of the rest, or at least such that these two products shall have a given ratio, for this condition does not make the problem any more difficult.

First, I suppose the thing done, and since so many lines are confusing, I may simplify matters by considering one of the given lines and one of those to be drawn (as, for example, AB and BC) as the principal lines, to which I shall try to refer all the others. Call the segment of the line AB between A and B, x , and call BC, y . Produce all the other given lines to meet these two (also produced if necessary) provided none is parallel to either of the principal lines. Thus, in the figure, the given lines cut AB in the points A, E, G, and cut BC in the points R, S, T.

Now, since all the angles of the triangle ARB are known,^[43] the ratio between the sides AB and BR is known.^[44] If we let $AB : BR = z : b$, since $AB = x$, we have $RB = \frac{bx}{z}$; and since B lies between C and R^[45],

we have $CR = y + \frac{bx}{z}$. (When R lies between C and B, CR is equal to $y - \frac{bx}{z}$, and when C lies between B and R, CR is equal to $-y + \frac{bx}{z}$.)

Again, the three angles of the triangle DRC are known,^[46] and therefore the ratio between the sides CR and CD is determined. Calling this ratio $z : c$, since $CR = y + \frac{bx}{z}$, we have $CD = \frac{cy}{z} + \frac{bx}{z^2}$. Then, since

^[43] Since BC cuts AB and AD under given angles.

^[44] Since the ratio of the sines of the opposite angles is known.

^[45] In this particular figure, of course.

^[46] Since CB and CD cut AD under given angles.

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the lines AB, AD, and EF are given in position, the distance from A to E is known. If we call this distance k , then $EB = k + x$; although $EB = k - x$ when B lies between E and A, and $E = -k + x$ when E lies between A and B. Now the angles of the triangle ESB being given, the ratio of BE to BS is known. We may call this ratio $z : d$.

Then $BS = \frac{dk + dx}{z}$ and $CS = \frac{zy + dk + dx}{z}$.^[47] When S lies between B and C we have $CS = \frac{zy - dk - dx}{z}$, and when C lies between B and S we have $CS = \frac{-zy + dk + dx}{z}$. The angles of the triangle FSC are known, and hence, also the ratio of CS to CF, or $z : e$. Therefore, $CF = \frac{ezy + dek + dex}{z^2}$. Likewise, AG or l is given, and $BG = l - x$. Also, in triangle BGT, the ratio of BG to BT, or $z : f$, is known. Therefore, $BT = \frac{fl - fx}{z}$ and $CT = \frac{zy + fl - fx}{z}$. In triangle TCH, the ratio of TC to CH, or $z : g$, is known,^[48] whence $CH = \frac{gzy + fgl - fgx}{z^2}$.

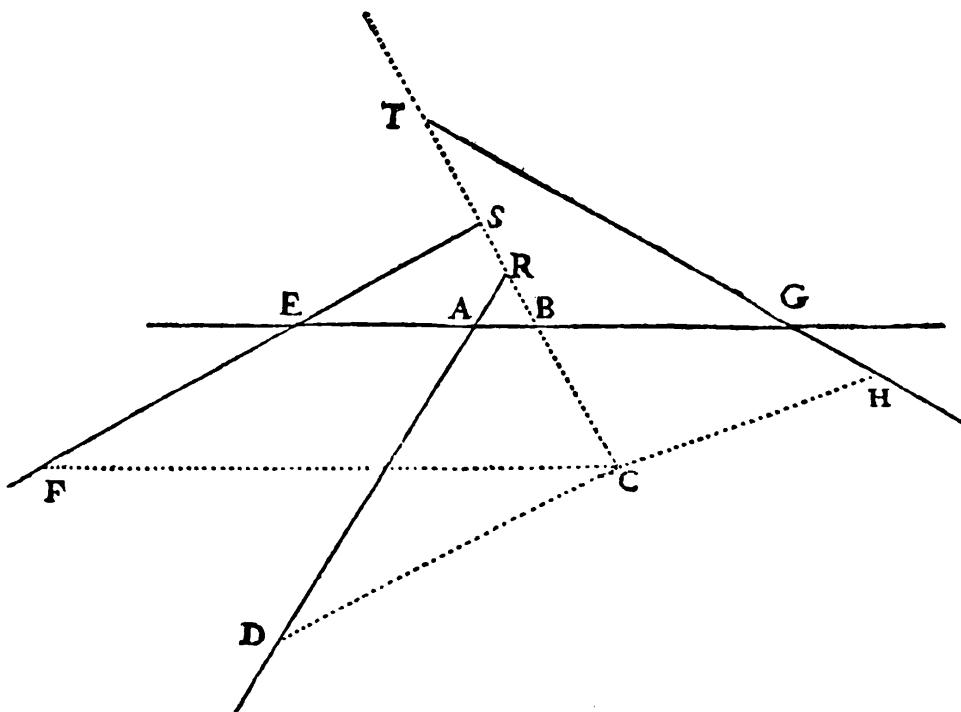
^[47] We have

$$\begin{aligned} CS &= y + BS \\ &= y + \frac{dk + dx}{z} \\ &= \frac{zy + dk + dx}{z}, \end{aligned}$$

and similarly for the other cases considered below.

The translation covers the first eight lines on the original page 312 (page 32 of this edition).

^[48] It should be noted that each ratio assumed has z as antecedent.



$C D$ fera $\frac{c y}{z} + \frac{b c x}{z z}$. Après cela pour ce que les lignes $A B$, $A D$, & $E F$ sont données par position, la distance qui est entre les points A & E est aussi donnée, & si on la nomme K , on aura $E B$ égal à $k + x$; mais ce seroit $k - x$, si le point B tomboit entre E & A ; & $-k + x$, si E tomboit entre A & B . Et pour ce que les angles du triangle $E S B$ sont tous donnés, la proportion de $B E$ à $B S$ est aussi donnée, & ie la pose comme z à d , si bien que $B S$ est $\frac{d k + d x}{z}$, & là toute $C S$ est $\frac{z y + d k + d x}{z}$; mais ce seroit $\frac{z y - d k - d x}{z}$, si le point S tomboit entre B & C ; & ce seroit $\frac{-z y + d k + d x}{z}$, si C tomboit entre B & S . De plus les trois angles du triangle $F S C$ sont donnés, & en suite la pro-

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proportion de CS à CF , qui soit comme de ζ à e , & la toute CF sera $\frac{ezy + dek + dex}{zz}$. En mesme façon AG que ie nomme l est donnée, & BG est $l - x$, & a cause du triangle BGT la proportion de BG à BT est aussi donnée, qui soit comme de ζ à f . & BT sera $\frac{fl - fx}{\zeta}$, & $CT \propto \frac{\zeta y + fl - fx}{z}$. Puis derechef la proportion de TC à CH est donnée, a cause du triangle UCH , & la posant comme de ζ à g , on aura $CH \propto \frac{fgzy + fg l - fg x}{zz}$.

Et ainsi vous voyés, qu'en tel nombre de lignes données par position qu'on puisse auoir, toutes les lignes tirées dessus du point C a angles donnés suivant la teneur de la question, se peuuent tousiours exprimer chascune par trois termes; dont l' vn est composé de la quantité inconnue y , multipliée , ou diuisee par quelque autre connue; & l'autre de la quantité inconnue x , aussi multipliée ou diuisee par quelque autre connuë , & le troisième d' vne quantité toute connuë. Excepté seulement si elles sont paralleles; oubien a la ligne AB , auquel cas le terme composé de la quantité x sera nul ; oubien a la ligne CB , auquel cas celuy qui est composé de la quantité y sera nul; ainsi qu'il est trop manifeste pour que ie m'arreste a l'expliquer. Et pour les signes $+$, & $-$, qui se iognent à ces termes, ils peuuent estre changés en toutes les façons imaginables.

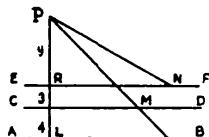
Puis vous voyés aussi, que multipliant plusieurs de ces lignes l' vne par l'autre, les quantités x & y , qui se trouuent dans le produit, n'y peuuent auoir que chascune autant de dimensions, qu'il y a eu de lignes, a l'explication

And thus you see that, no matter how many lines are given in position, the length of any such line through C making given angles with these lines can always be expressed by three terms, one of which consists of the unknown quantity y multiplied or divided by some known quantity; another consisting of the unknown quantity x multiplied or divided by some other known quantity; and the third consisting of a known quantity.^[49] An exception must be made in the case where the given lines are parallel either to AB (when the term containing x vanishes), or to CB (when the term containing y vanishes). This case is too simple to require further explanation.^[50] The signs of the terms may be either + or — in every conceivable combination.^[51]

You also see that in the product of any number of these lines the degree of any term containing x or y will not be greater than the number of lines (expressed by means of x and y) whose product is found. Thus, no term will be of degree higher than the second if two lines be multiplied together, nor of degree higher than the third, if there be three lines, and so on to infinity.

^[49] That is, an expression of the form $ax + by + c$, where a, b, c , are any real positive or negative quantities, integral or fractional (not zero, since this exception is considered later).

^[50] The following problem will serve as a very simple illustration: Given three parallel lines AB, CD, EF, so placed that AB is distant 4 units from CD, and CD is distant 3 units from EF; required to find a point P such that if PL, PM, PN



be drawn through P, making angles of 90° , 45° , 30° , respectively, with the parallels. Then $\overline{PM}^2 = PL \cdot PN$.

Let $PR = y$, then $PN = 2y$, $PM = \sqrt{2}(y+3)$, $PL = y+7$. If $\overline{PM}^2 = PN \cdot PL$, we have $[\sqrt{2}(y+3)]^2 = 2y(y+7)$, whence $y = 9$. Therefore, the point P lies on the line XY parallel to EF and at a distance of 9 units from it. Cf. Rabuel, p. 79.

^[51] Depending, of course, upon the relative positions of the given lines.

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Furthermore, to determine the point C, but one condition is needed, namely, that the product of a certain number of lines shall be equal to, or (what is quite as simple), shall bear a given ratio to the product of certain other lines. Since this condition can be expressed by a single equation in two unknown quantities,^[52] we may give any value we please to either x or y and find the value of the other from this equation. It is obvious that when not more than five lines are given, the quantity x , which is not used to express the first of the lines can never be of degree higher than the second.^[53]

Assigning a value to y , we have $x^2 = \pm ax \pm b^2$, and therefore x can be found with ruler and compasses, by a method already explained.^[54] If then we should take successively an infinite number of different values for the line y , we should obtain an infinite number of values for the line x , and therefore an infinity of different points, such as C, by means of which the required curve could be drawn.

This method can be used when the problem concerns six or more lines, if some of them are parallel to either AB or BC, in which case

^[52] That is, an indeterminate equation. "De plus, à cause que pour determiner le point C, il n'y a qu'une seule condition qui soit requise, à sçavoir que ce qui est produit par la multiplication d'un certain nombre de ces lignes soit égal, ou (ce qui n'est de rien plus mal-aisé) ait la proportion donnée, à ce qui est produit par la multiplication des autres; on peut prendre à discretion l'une des deux quantitez inconnues x ou y , & chercher l'autre par cette Equation." Such variations in the texts of different editions are of no moment, but are occasionally introduced as matters of interest.

^[53] Since the product of three lines bears a given ratio to the product of two others and a given line, no term can be of higher degree than the third, and therefore, than the second in x .

^[54] See pages 13, et seq.

cation desquelles elles seruent, qui ont esté ainsi multipliées: en sorte qu'elles n'auront iamais plus de deux dimensions, en ce qui ne sera produit que par la multiplication de deux lignes; ny plus de trois, en ce qui ne sera produit que par la multiplication de trois, & ainsi a l'infini .

De plus, a cause que pour determiner le point C, il Comme^{er}
on trouve n'y a qu'vne seule condition qui soit requise , à sçauoir que ce
proble-
me est que ce qui est produit par la multiplication d vn certain plan, lors-
qu'il n'est
point nombre de ces lignes soit esgal, ou (ce qui n'est de rien plus malayse) ait la proportion donnée , à ce qui est produ^tuit par la multiplication des autres; on peut prendre a discretion l vne des deux quantités inconnues x ou y , & en plus de
5 lignes. chercher l autre par cete Equation. en laquelle il est euident que lorsque la question n'est point proposée en plus de cinq lignes, la quantité x qui ne fert point a l expression de la premiere peut tousiours n'y auoir que deux dimensions. de façon que prenant vne quantité connue pour y, il ne restera que $x x \infty +$ ou $-ax+$ ou $-bb.$ & ainsi on pourra trouuer la quantité x avec la reigle & le compas, en la facon tantost expliquée. Mesme prenant successiuement infinies diuerses grandeurs pour la ligne y, on en trounera aussy infinies pour la ligne x, & ainsi on aura vne infinité de diuers poins , tels que celuy qui est marqué C , par le moyen desquels on descrira la ligne courbe demandée.

Il se peut faire aussy, la question estant proposée en six, ou plus grand nombre de lignes; s'il y en a entre les données, qui soient paralleles a BA, ou BC , que l vne des deux quantités x ou y n'ait que deux dimensions en

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l'Equation, & ainsi qu'on puisse trouuuer le point C avec la reigle & le compas. Mais au contraire si elles sont toutes paralleles , encore que la question ne soit proposée qu'en cinq lignes, ce point C ne pourra ainsi estre trouué, a cause que la quantité x ne se trouuant point en toute l'Equation, il ne sera plus permis de prendre vne quantité connue pour celle qui est nommée y , mais ce sera elle qu'il faudra chercher. Et pource quelle aura trois dimensions, on ne la pourra trouuer qu'en tirant la racine d'une Equation cubique. ce qui ne se peut généralement faire sans qu'on y employe pour le moins vne section conique. Et encore qu'il y ait iusques a neuf lignes données, pourvûqu'elles ne soient point toutes paralleles, on peut toufiours faire que l'Equation ne monte que iusques au quarré de quarré. au moyen de quoy on la peut aussy toufiours resoudre par les sections coniques, en la façon que iexpliqueray cy après. Et encore qu'il y en ait iusques a treize , on peut toufiours faire qu'elle ne monte que iusques au quarré de cube. en suite de quoy on la peut resoudre par le moyen d'une ligne , qui n'est que d'un degré plus composée que les sections coniques, en la façon que iexpliqueray aussy cy après. Et cecy est la première partie de ce que iauois icy a demontrer ; mais avant que ie passe a la seconde il est besoin que ie dise quelque chose en general de la nature des lignes courbes.

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either x or y will be of only the second degree in the equation, so that the point C can be found with ruler and compasses.

On the other hand, if the given lines are all parallel even though a question should be proposed involving only five lines, the point C cannot be found in this way. For, since the quantity x does not occur at all in the equation, it is no longer allowable to give a known value to y . It is then necessary to find the value of y .^[55] And since the term in y will now be of the third degree, its value can be found only by finding the root of a cubic equation, which cannot in general be done without the use of one of the conic sections.^[56]

And furthermore, if not more than nine lines are given, not all of them being parallel, the equation can always be so expressed as to be of degree not higher than the fourth. Such equations can always be solved by means of the conic sections in a way that I shall presently explain.^[57]

Again, if there are not more than thirteen lines, an equation of degree not higher than the sixth can be employed, which admits of solution by means of a curve just one degree higher than the conic sections by a method to be explained presently.^[58]

This completes the first part of what I have to demonstrate here, but it is necessary, before passing to the second part, to make some general statements concerning the nature of curved lines.

^[55] That is, to solve the equation for y .

^[56] See page 84.

^[57] See page 107.

^[58] This line of reasoning may be extended indefinitely. Briefly, it means that for every two lines introduced the equation becomes one degree higher and the curve becomes correspondingly more complex.